Effects of anisotropy on thermal entanglement

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We study the thermal entanglement in the two-qubit anisotropic XXZ model and the Heisenberg model with Dzyaloshinski-Moriya (DM) interactions. The DM interaction is another kind of anisotropic antisymmetric exchange interaction. The effects of these two kinds of anisotropies on the thermal entanglement are studied in detail for both the antiferromagnetic and ferromagnetic cases.

Recently the concept of thermal entanglement was introduced and studied within one-dimensional isotropic Heisenberg model [1]. The state of the system described by the Hamiltonian H at thermal equilibrium is $\rho(T) = \exp\left(-\frac{H}{kT}\right)/Z$, where $Z = \text{Tr}\left[\exp\left(-\frac{H}{kT}\right)\right]$ is the partition function and k is the Boltzmann's constant. As $\rho(T)$ represents a thermal state, the entanglement in the state is called the thermal entanglement [1].

For two-qubit isotropic Heisenberg model there exists thermal entanglement for the antiferromagnetic case and no thermal entanglement for the ferromagnetic case [1]. While for the XY model the thermal entanglement appears for both the antiferromagnetic and ferromagnetic cases [2]. It is known that the isotropic Heisenberg model and the XY model are special cases of the anisotropic Heisenberg model (see Eq.(3)). So it is worth to study the thermal entanglement in the anisotropic models and see the role of anisotropic parameters. In this paper we consider two types of anisotropy and study the effects of them on the thermal entanglement. Both the antiferromagnetic and ferromagnetic cases are considered.

First we briefly review a measure of entanglement, the concurrence [3]. Let ρ_{12} be the density matrix of a pair of qubits 1 and 2. The density matrix can be either pure or mixed. The concurrence corresponding to the density matrix is defined as

$$C_{12} = \max\left\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\right\}, \qquad (1)$$

where the quantities $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are the square roots of the eigenvalues of the operator

$$\varrho_{12} = \rho_{12}(\sigma_{1y} \otimes \sigma_{2y})\rho_{12}^*(\sigma_{1y} \otimes \sigma_{2y}). \qquad (2)$$

The opeators $\sigma_{jy}(j=1,2)$ are the usual Pauli operators for the qubit j. The concurrence $C_{12} = 0$ corresponds to an unentangled state and $C_{12} = 1$ corresponds to a maximally entangled state.

We consider the two-qubit anisotropic XXZ Heisenberg model [4,5]

$$
\begin{aligned}
H &= \frac{J}{2}\left(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \Delta\sigma_{1z}\sigma_{2z}\right) \\
&= J\left(\sigma_{1+}\sigma_{2-} + \sigma_{1-}\sigma_{2+}\right) + \frac{J\Delta}{2}\sigma_{1z}\sigma_{2z}, \qquad (3)
\end{aligned}
$$

where the coupling constants $J > 0$ corresponds to the antiferromagnetic case and $J < 0$ the ferromagnetic case. The operators $\sigma_{j\pm} = \frac{1}{2}\left(\sigma_{jx} \pm i\sigma_{jy}\right)(j=1,2)$. The XXZ model was initiated by Bethe for the case $\Delta = \pm 1$ in 1931 [4] and has been studied for $\Delta \neq \pm 1$ since 1959 [5].

The eigenvalues and eigenvectors of H are easily obtained as

$$
\begin{aligned}
H|00\rangle &= \frac{J\Delta}{2}|00\rangle, \quad H|11\rangle = \frac{J\Delta}{2}|11\rangle, \\
H|\Psi^\pm\rangle &= \left(-\frac{J\Delta}{2} \pm J\right)|\Psi^\pm\rangle, \qquad (4)
\end{aligned}
$$

where $|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are maximally entangled states and $|0\rangle$ $(|1\rangle)$ denotes the ground (excited) state of a two-level particle.

In the standard basis, $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, the density matrix $\rho(T)$ is written as $(k=1)$

$$\rho(T) = \frac{1}{2(e^{\frac{J\Delta}{2T}}\cosh\frac{J}{T} + e^{-\frac{J\Delta}{2T}})} \qquad (5)$$

$$
\times
\begin{pmatrix}
e^{-\frac{J\Delta}{2T}} & 0 & 0 & 0 \\
0 & e^{\frac{J\Delta}{2T}}\cosh\frac{J}{T} & -e^{\frac{J\Delta}{2T}}\sinh\frac{J}{T} & 0 \\
0 & -e^{\frac{J\Delta}{2T}}\sinh\frac{J}{T} & e^{\frac{J\Delta}{2T}}\cosh\frac{J}{T} & 0 \\
0 & 0 & 0 & e^{-\frac{J\Delta}{2T}}
\end{pmatrix}.
$$

The square roots of the four eigenvalues of the density matrix ϱ_{12} are

$$
\begin{aligned}
\lambda_1 &= \lambda_2 = \frac{e^{-\frac{J\Delta}{T}}}{2(\cosh\frac{J}{T} + e^{-\frac{J\Delta}{T}})}, \\
\lambda_3 &= \frac{e^{\frac{J}{T}}}{2(\cosh\frac{J}{T} + e^{-\frac{J\Delta}{T}})}, \\
\lambda_4 &= \frac{e^{-\frac{J}{T}}}{2(\cosh\frac{J}{T} + e^{-\frac{J\Delta}{T}})}. \qquad (6)
\end{aligned}
$$

Which is the largest eigenvalue depends on the value of anisotropic parameter Δ and sign of J. For antiferromagnetic case $(J > 0)$ the largest eigenvalue is λ_1 when $\Delta \leq -1$ and λ_3 when $\Delta > -1$. Therefore the concurrences are given by

$$C_{AFM}(\Delta) = 0 \text{ for } \Delta \leq -1,$$
$$C_{AFM}(\Delta) = \max\left(\frac{\sinh(\frac{J}{T}) - e^{-\frac{J\Delta}{T}}}{\cosh\frac{J}{T} + e^{-\frac{J\Delta}{T}}}, 0\right)$$
$$\text{for } \Delta > -1. \tag{7}$$

When $\Delta = 1$, the anisotropic model becomes the isotropic model, and Eq.(7) reduces to

$$C_{AFM}(1) = \max\left(\frac{e^{\frac{2J}{T}} - 3}{e^{\frac{2J}{T}} + 3}, 0\right) \tag{8}$$

which is obtained in Ref. [1]. From the above equation we know that when the temperature is larger than the critical temperature $T_C = \frac{2J}{\ln 3}$ the thermal entanglement disappears. For the anisotropic model the critical temperature T_C is determined by the nonlinear equation

$$\sinh(\frac{J}{T}) = e^{-\frac{J\Delta}{T}}. \tag{9}$$

For ferromagnetic case ($J < 0$) the largest eigenvalue is λ_4 when $\Delta < 1$ and λ_1 when $\Delta \geq 1$. Therefore the concurrences are

$$C_{FM}(\Delta) = 0 \text{ for } \Delta \geq 1,$$
$$C_{FM}(\Delta) = \max\left(\frac{\sinh(\frac{|J|}{T}) - e^{\frac{|J|\Delta}{T}}}{\cosh\frac{|J|}{T} + e^{\frac{|J|\Delta}{T}}}, 0\right)$$
$$\text{for } \Delta < 1. \tag{10}$$

From the above equation we see that no thermal entanglement for the ferromagnetic isotropic Heisenberg model ($\Delta = 1$). The critical temperature is given by the equation

$$\sinh(\frac{|J|}{T}) = e^{\frac{|J|\Delta}{T}}. \tag{11}$$

From Eqs.(7) and (10) it is find that the thermal entanglement are same when $\Delta = 0$. That is to say, the entanglement exists in the antiferromagnetic and ferromagnetic models at the same time. The Heisenberg Hamiltonian with $\Delta = 0$ is just the quantum XY model. The thermal entanglement this model is discussed in a recent paper [2]. From Eqs.(7) and (10) we also see that the concurrences satisfy $C_{AFM}(\Delta) = C_{FM}(-\Delta)$.

We numerically solved Eqs.(9) and (11) and the results are shown in Fig.1. For the antiferromagnetic case we observe that the critical temperature T_C increases as the anisotropic parameter Δ increases. Oppositely T_C decreases as Δ increases for the ferromagnetic case. Of course the critical temperatures are same when $\Delta = 0$, which corresponding to the XY model.

Fig.2(a) gives a plot of the concurrence as a function of temperature for the antiferromagnetic case. It shows

that the concurrences are 1 for different anisotropic parameters when $T = 0$. In these cases the ground state is $|\Psi^-\rangle$, which is the maximally entangled state and the corresponding concurrences are 1. As the temperature increases, the concurrence decreases due to the mixing of other states with the maximally entangled state. Again we see that T_C increases as Δ increases.
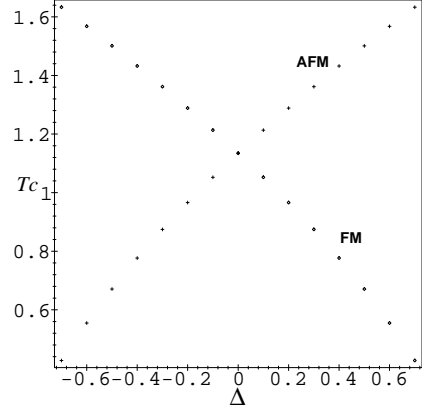

FIG. 1. The critical temperature as a function of anisotropic parameter Δ for both the antiferromagnetic and ferromagnetic cases. The parameter $J = 1$.

Another kind of anisotropy is the DM anisotropic antisymmetric interaction which arises from spin-orbit coupling [6,7]. Now we consider the Heisenberg model with DM interaction

$$H_{DM} = \frac{J}{2}[(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \Delta\sigma_{1z}\sigma_{2z})$$
$$+\vec{D}\cdot(\vec{\sigma}_1 \times \vec{\sigma}_2)], \tag{12}$$

where $\vec{D}$ is the DM vector coupling. To see the effect of the anisotropic parameter $\vec{D}$ we choose $\vec{D} = D\vec{z}$ and $\Delta = 0$. Then the Hamiltonian H_{DM} becomes

$$H_{DM} = \frac{J}{2}[\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + D(\sigma_{1x}\sigma_{2y} - \sigma_{1y}\sigma_{2x})]$$
$$= J[(1 + iD)\sigma_{1+}\sigma_{2-} + (1 - iD)\sigma_{1-}\sigma_{2+}]. \tag{13}$$

The eigenvalues and eigenvectors of H_{DM} are given by

$$H_{DM}|00\rangle = 0, H_{DM}|11\rangle = 0,$$
$$H_{DM}|\pm\rangle = \pm J\sqrt{1 + D^2}|\pm\rangle, \tag{14}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm e^{i\theta}|10\rangle\right)$ and $\theta = \arctan D$.

In the standard basis, the density matrix $\rho(T)$ is given by

$$\rho(T) = \frac{1}{2(\cosh\frac{J\sqrt{1+D^2}}{T} + 1)} \qquad (15)$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\frac{J\sqrt{1+D^2}}{T} & -e^{-i\theta}\sinh\frac{J\sqrt{1+D^2}}{T} & 0 \\ 0 & -e^{i\theta}\sinh\frac{J\sqrt{1+D^2}}{T} & \cosh\frac{J\sqrt{1+D^2}}{T} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The square roots of the four eigenvalues of the density matrix ϱ_{12} are

$$\lambda_1 = \lambda_2 = \frac{1}{2(\cosh\frac{J\sqrt{1+D^2}}{T} + 1)},$$

$$\lambda_3 = \frac{e^{\frac{J\sqrt{1+D^2}}{T}}}{2(\cosh\frac{J\sqrt{1+D^2}}{T} + 1)},$$

$$\lambda_4 = \frac{e^{-\frac{J\sqrt{1+D^2}}{T}}}{2(\cosh\frac{J\sqrt{1+D^2}}{T} + 1)}. \qquad (16)$$

We see that the four eigenvalues are independent on the angle θ. From the eigenvalues we observe that for both antiferromagnetic and ferromagnetic cases the concurrences are given by

$$C = \max\left(\frac{\sinh\frac{|J|\sqrt{1+D^2}}{T} - 1}{\cosh\frac{|J|\sqrt{1+D^2}}{T} + 1}, 0\right). \qquad (17)$$

We see that the entanglement does not depend on the sign of the anisotropic parameter D.

The critical temperature is given by

$$T_C = \frac{|J|\sqrt{1+D^2}}{\arcsin h(1)} \approx 1.1346\sqrt{1+D^2}|J|. \qquad (18)$$

Obviously the critical temperature increases with the increase of the absolute value of the anisotropic parameter D, which can also be seen in Fig.2(b). The ground state of the system is $|+\rangle$ or $|-\rangle$ no matter what the anisotropic parameter D is. Fig.2(b) shows that the concurrences are 1 when $T = 0$, which is due to the maximally entangled ground state $|\pm\rangle$. When the temperature is larger than the critical temperature the thermal entanglement disappears.
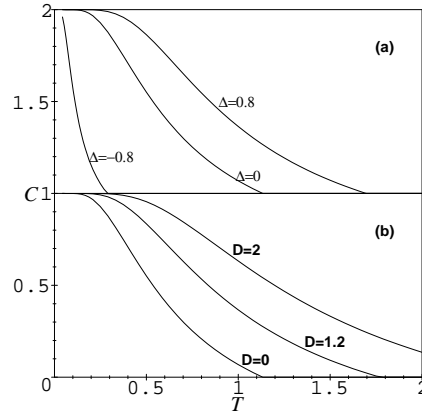

FIG. 2. The concurrences as a function of temperature. (a) For anisotropic antiferromagnetic Heisenberg model; (b)For Heisenberg model with DM interaction. The parameter $|J| = 1$.

In conclusion we have studied the effect of two kinds of anisotropy on the thermal entanglement in the anisotropic XXZ model and the Heisenberg model with DM interaction. For the XXZ model it is shown that the thermal entanglement exist or not depends on both the anisotropic parameters and the sign of exchange constants J. The thermal entanglement are same for the antiferromagnetic and ferromagnetic Heisenberg model with DM interaction. While in the XXZ model the thermal entanglements are different for the antiferromagnetic and ferromagnetic cases. In this paper we restrict ourselves to the two-qubit case. It is a good challenge to study thermal entanglement in the multi-qubit anisotropic models.

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